

Characteristic matrix of covering and its application to boolean matrix decomposition and axiomatization

Shiping Wang^a, William Zhu^{b,*}, Qingxin Zhu^a, Fan Min^b

^a*School of Computer Science and Engineering,
University of Electronic Science and Technology of China, Chengdu 611731, China*

^b*Lab of Granular Computing,
Zhangzhou Normal University, Zhangzhou 363000, China*

Abstract

Covering is an important type of data structure while covering-based rough sets provide an efficient and systematic theory to deal with covering data. In this paper, we use boolean matrices to represent and axiomatize three types of covering approximation operators. First, we define two types of characteristic matrices of a covering which are essentially square boolean ones, and their properties are studied. Through the characteristic matrices, three important types of covering approximation operators are concisely equivalently represented. Second, matrix representations of covering approximation operators are used in boolean matrix decomposition. We provide a sufficient and necessary condition for a square boolean matrix to decompose into the boolean product of another one and its transpose. And we develop an algorithm for this boolean matrix decomposition. Finally, based on the above results, these three types of covering approximation operators are axiomatized using boolean matrices. In a word, this work borrows extensively from boolean matrices and present a new view to study covering-based rough sets.

Keywords: Covering, Rough sets, Boolean matrix, Characteristic matrix, Approximation operator.

1. Introduction

As a widely used form of data representation, coverings most commonly appear in incomplete information/decision systems based on symbol data [3, 4, 24], numeric and fuzzy data [10, 11, 22]. Covering-based rough set theory [23, 29] is an efficient tool to deal with covering data. In recent years, it has attracted much research interest with a series of significant problems proposed. For example, different approximation models have been constructed [26, 28, 34], covering reducts problems have been defined [6, 21, 27, 33], generalization works have been conducted [5, 7, 12, 31] and combinations with other theories have been made [9, 19, 25].

*Corresponding author.

Email address: williamfengzhu@gmail.com (William Zhu)

Specifically, axiomatization of covering approximation operators has been a hot issue. For example, Zhu and Wang [33, 34] proposed the reducible element to axiomatize the covering lower approximation operator. Following by Zhu and Wang' work, Zhang et al. [30] axiomatized three pairs of covering approximation operators. Liu and Sai [17] constructed an axiom of a pair of covering approximation operators from the viewpoint of operator theory. Unfortunately, as an efficient tool for computable models, matrices have been seldom used in covering-based rough sets. However, increasing achievements have been made in representing and axiomatizing classical and fuzzy rough sets using matrices [15, 16, 18]. Naturally, this motivates us to represent and axiomatize covering-based rough sets using boolean matrices.

Boolean matrix decomposition has not only important practical meaning, but also profound theoretical significance. In application, it has been widely used in data mining [2], role engineering [20] and machine learning [8], and so on. In theory, boolean matrix decomposition such as boolean rank and boolean multiplication has attracted much research interest [1, 13]. However, in general, boolean matrix decomposition is NP-hard, hence it is often addressed as an optimization problem.

In this paper, we represent three pairs of covering approximation operators using boolean matrices, and the representation in turn is used in boolean matrix decomposition and axiomatization. First, we define two types of characteristic matrices of a covering and use them to concisely represent three pairs of covering approximation operators. Second, through the matrix representation of covering upper approximation operator, we present a sufficient and necessary condition for a square boolean matrix to decompose into the boolean product of another boolean matrix and its transpose. And an algorithm to complement this decomposition is designed. Third, using the sufficient and necessary condition of boolean matrix decomposition, we axiomatize these three types of covering approximation operators.

The rest of this paper is arranged as follows. Section 2 reviews some fundamental concepts related to covering-based rough sets. In Section 3, we present two types of characteristic matrices of a covering and use them to represent three types of covering approximation operators. Section 4 exhibits a sufficient and necessary condition, and an algorithm for boolean matrix decomposition. In Section 5, we axiomatize these three types of covering approximation operators through boolean matrix decomposition. Section 6 concludes this paper and points out further work.

2. Basic definitions

This section recalls some fundamental definitions and existing results concerning covering-based rough sets.

Definition 1. (*Covering [33]*) Let U be a finite universe of discourse and \mathbf{C} a family of subsets of U . If none of subsets in \mathbf{C} is empty and $\bigcup \mathbf{C} = U$, then \mathbf{C} is called a covering of U .

Neighborhoods are important concepts in rough sets, and they can describe the maximal and minimal dependence to an object.

Definition 2. (*Indiscernible neighborhood and neighborhood [32]*) Let \mathbf{C} be a covering of U and $x \in U$. $I_{\mathbf{C}}(x) = \bigcup\{K \in \mathbf{C} | x \in K\}$ and $N_{\mathbf{C}}(x) = \bigcap\{K \in \mathbf{C} | x \in K\}$ are called the indiscernible neighborhood and neighborhood of x with respect to \mathbf{C} , respectively. When there is no confusion, we omit the subscript \mathbf{C} .

Neighborhood granulation derived from coverings is a basic unit to characterize data, and leads to neighborhood-based decision systems, where neighborhood-based approximation operators have been used extensively in symbolic or/and numerical attribute reduction [10]. In this paper, we study the following three types of lower and upper approximation operators.

Definition 3. (*Approximation operators [23, 32]*) Let \mathbf{C} be a covering of U . For all $X \subseteq U$,
 $SH_{\mathbf{C}}(X) = \bigcup\{K \in \mathbf{C} | K \cap X \neq \emptyset\}$, $SL_{\mathbf{C}}(X) = [SH_{\mathbf{C}}(X^c)]^c$,
 $IH_{\mathbf{C}}(X) = \{x \in U | N(x) \cap X \neq \emptyset\}$, $IL_{\mathbf{C}}(X) = \{x \in U | N(x) \subseteq X\}$,
 $XH_{\mathbf{C}}(X) = \bigcup\{N(x) | N(x) \cap X \neq \emptyset\}$, $XL_{\mathbf{C}}(X) = \bigcup\{N(x) | N(x) \subseteq X\}$,
are called the second, fifth, and sixth upper and lower approximations of X with respect to \mathbf{C} , respectively. When there is no confusion, we omit the subscript \mathbf{C} .

In real world applications, much knowledge is redundant, therefore it is necessary to remove the redundancy and keep the essence. For example, the reducible element can deal with the knowledge redundancy in rule learning [6].

Definition 4. (*Reducible element [33]*) Let \mathbf{C} be a covering of U and $K \in \mathbf{C}$. If K is a union of some sets in $\mathbf{C} - \{K\}$, then K is called reducible; otherwise K is called irreducible. All irreducible elements of \mathbf{C} is called reduct of \mathbf{C} , denoted as $Reduct(\mathbf{C})$.

3. Matrix representation of covering approximation operators

In this section, we define the matrix representation a family of subsets of a set, and then propose two types of characteristic matrices of a covering. Through these two characteristic matrices of a covering, we represent three types of covering approximation operators.

3.1. Matrix representation of family of subsets of a set

This section represents a family of subsets of a set using a zero-one matrix, called boolean matrix. Using this matrix, families of subsets are connected with binary relations and their further properties are found.

Definition 5. Let $\mathbf{F} = \{F_1, \dots, F_m\}$ be a family of subsets of a finite set $U = \{x_1, \dots, x_n\}$. We define $M_{\mathbf{F}} = (m_{ij})_{n \times m}$ as follows:

$$m_{ij} = \begin{cases} 1, & x_i \in F_j, \\ 0, & x_i \notin F_j. \end{cases}$$

$M_{\mathbf{F}}$ is called a matrix representation of \mathbf{F} , or called a matrix representing \mathbf{F} .

The following example shows that different matrices can be used to represent the same family of subsets of a set.

Example 1. Let $U = \{a, b, c, d, e\}$ and $\mathbf{F} = \{\{a, b, c\}, \{b, d\}, \{c, d\}\}$. Then M_1 and M_2 are matrices representing \mathbf{F} .

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are matrices representing the same family of subsets of a set, however it is interesting that the boolean product of one matrix and its transpose is unique once a sort of elements of the universe is given.

Proposition 1. Let $\mathbf{F} = \{F_1, \dots, F_m\}$ be a family of subsets of $U = \{x_1, \dots, x_n\}$ and M_1, M_2 matrices representing \mathbf{F} . Then $M_1 \cdot M_1^T = M_2 \cdot M_2^T$, where $M \cdot M^T$ is the boolean product of M and its transpose M^T .

PROOF. Since M_1 and M_2 are matrices representing \mathbf{F} , then M_1 can be transformed into M_2 through list exchanges. Hence we only need to prove $M_1 \cdot M_1^T = M_2 \cdot M_2^T$ when $M_1 = (a_1, \dots, a_i, \dots, a_j, \dots, a_m)$ and $M_2 = (a_1, \dots, a_j, \dots, a_i, \dots, a_m)$ for $1 \leq i < j \leq m$, where a_k is a n -dimensional column vector. Thus $M_1 \cdot M_1^T = (a_1, \dots, a_m) \cdot \begin{pmatrix} a_1^T \\ \vdots \\ a_i^T \\ \vdots \\ a_j^T \\ \vdots \\ a_m^T \end{pmatrix} = \bigvee_{k=1}^m (a_k \cdot a_k^T) = M_2 \cdot M_2^T$.

The following example is provided to illustrate the uniqueness of the boolean product of any matrix representing a covering and its transpose.

Example 2. As shown in Example 1, $M_1 \cdot M_1^T = M_2 \cdot M_2^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Suppose R is a relation on $U = \{x_1, \dots, x_n\}$, then its relational matrix $M_R = (m_{ij})_{n \times n}$ is defined as follows:

$$m_{ij} = \begin{cases} 1, & (x_i, x_j) \in R, \\ 0, & (x_i, x_j) \notin R. \end{cases}$$

Conversely, for any n -by- n boolean matrix M , there exists a relation R such that $M = M_R$; we say R is induced by M .

According to the above, there is a one-to-one correspondence between binary relations on U and $|U| \times |U|$ boolean matrices. Then the connecting between matrices representing families of subsets and binary relations is built.

Proposition 2. *Let \mathbf{F} be a family of subsets of U . There exists a symmetric relation R such that $M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T$ is the relational matrix of R .*

PROOF. We only need to prove that $M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T$ is a symmetric matrix. It is straightforward since $(M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T)^T = (M_{\mathbf{F}}^T)^T \cdot M_{\mathbf{F}}^T = M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T$.

The square matrix, the boolean product of a matrix representing a family of subsets of a universe and its transpose, is regarded as a whole and occupies idempotence.

Proposition 3. *Let $\mathbf{F} = \{F_1, \dots, F_m\}$ be a family of subsets of U . If for all $1 \leq i < j \leq m$, $F_i \cap F_j = \emptyset$, then $(M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T)^2 = M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T$.*

PROOF. Denote $M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot (a_1^T, \dots, a_n^T) = (t_{ij})_{n \times n}$ and $(M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T)^2 = (s_{ij})_{n \times n}$. Since for all $1 \leq i < j \leq m$, $F_i \cap F_j = \emptyset$, then $t_{ij} = a_i \cdot a_j^T = \begin{cases} 1, & \exists F \in \mathbf{F}, \text{ s.t. } x_i, x_j \in F, \\ 0, & \text{otherwise.} \end{cases}$. If $t_{ij} = 1$, then $s_{ij} = \bigvee_{k=1}^n (t_{ik} \wedge t_{kj}) \geq t_{ij} \wedge t_{jj} = 1$, which implies $s_{ij} = 1$. If $t_{ij} = 0$ and $i = j$, then $x_i \notin \bigcup \mathbf{F}$ and $a_i = [0, \dots, 0]$, which implies $s_{ij} = 0$. If $t_{ij} = 0$ and $i \neq j$, we need to prove $s_{ij} = 0$. In fact, if $s_{ij} = \bigvee_{k=1}^n (t_{ik} \wedge t_{kj}) = 1$, then there exists $k_0 \in \{1, \dots, n\}$ such that $t_{ik_0} = t_{jk_0} = 1$. Thus there exist $F_g, F_h \in \mathbf{F}$ such that $x_i, x_{k_0} \in F_g$ and $x_j, x_{k_0} \in F_h$. Since $t_{ij} = 0$, then $F_g \neq F_h$. Therefore, $x_{k_0} \in F_g \cap F_h$, which is contradictory with $F_i \cap F_j = \emptyset$ for all $i, j \in \{1, \dots, n\}$ and $i \neq j$.

Corollary 1. *Let $\mathbf{F} = \{F_1, \dots, F_m\}$ be a family of subsets of U . If for all $1 \leq i < j \leq m$, $F_i \cap F_j = \emptyset$, then there exists a transitive relation R such that $M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T$ is the relational matrix of R .*

Corollary 2. *If \mathbf{P} is a partition of U , then $(M_{\mathbf{P}} \cdot M_{\mathbf{P}}^T)^2 = M_{\mathbf{P}} \cdot M_{\mathbf{P}}^T$.*

3.2. Type-1 characteristic matrix of covering

In this section, type-1 characteristic matrix of a covering is defined, and then connections between coverings and binary relations are established.

Definition 6. (Type-1 characteristic matrix of covering) *Let \mathbf{C} be a covering of U . Then $M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T$ is called type-1 characteristic matrix of \mathbf{C} , denoted as $\Gamma(\mathbf{C})$.*

Properties of type-1 characteristic matrix of a covering are studied. In fact, its elements on the main diagonal are equal to one.

Proposition 4. *Let \mathbf{F} be a family of subsets of U and $\emptyset \notin \mathbf{F}$. \mathbf{F} is a covering iff all the elements on the main diagonal of $M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T$ are one.*

PROOF. Denote $M_{\mathbf{F}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, where a_i is a m -dimensional row vector. Denote

$$T_{\mathbf{F}} = (t_{ij})_{n \times n} = M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T = (t_{ij})_{n \times n}.$$

(\implies): Since \mathbf{F} is a covering of U , then $a_i \neq [0, \dots, 0]$ for all $i \in \{1, \dots, m\}$. Hence $t_{ii} = \bigvee_{k=1}^m (a_{ik} \wedge a_{ki}) = \bigvee_{k=1}^m (a_{ik} \wedge a_{ik}) = 1$.

(\impliedby): If \mathbf{F} is not a covering of U , then we suppose $x_i \in U - \bigcup \mathbf{F}$. Thus $a_i = [0, \dots, 0]$ implies $t_{ii} = 0$, which is contradictory that all the elements on the main diagonal of $M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T$ are one.

According to the above property of type-1 characteristic matrix of a covering, the relationship between coverings and reflexive relations is established.

Corollary 3. *Let \mathbf{F} be a family of subsets of U and $\emptyset \notin \mathbf{F}$. \mathbf{F} is a covering iff there exists a reflexive relation R such that $M_{\mathbf{F}} \cdot M_{\mathbf{F}}^T$ is the relational matrix of R .*

Since type-1 characteristic matrix of a covering is a square boolean matrix, it can induce a binary relation. The following proposition represents the relation through covering blocks.

Proposition 5. *Let \mathbf{C} be a covering of U and $R_{\mathbf{C}}$ the relation induced by $\Gamma(\mathbf{C})$. For all $x_i, x_j \in U$, $(x_i, x_j) \in R_{\mathbf{C}}$ iff there exists $C \in \mathbf{C}$ such that $x_i, x_j \in C$.*

PROOF. For all $x_i, x_j \in U$, $(x_i, x_j) \in R_{\mathbf{C}}$
 $\Leftrightarrow t_{ij} = \bigvee_{k=1}^m (a_{ik} \wedge a_{kj}) = \bigvee_{k=1}^m (a_{ik} \wedge a_{jk}) = 1$
 $\Leftrightarrow a_i \wedge a_j \neq [0, \dots, 0]$
 \Leftrightarrow there exists $C \in \mathbf{C}$ such that $x_i, x_j \in C$.

It is interesting that type-1 characteristic matrix of a covering is the relational matrix of the relation induced by indiscernible neighborhoods of the covering.

Theorem 1. *Let \mathbf{C} be a covering of U and $R_{\mathbf{C}}$ the relation induced by $\Gamma(\mathbf{C})$. For all $x, y \in U$, $(x, y) \in R_{\mathbf{C}}$ iff $y \in I_{\mathbf{C}}(x) = \bigcup \{K \in \mathbf{C} | x \in K\}$.*

PROOF. (\implies): According to Proposition 5, if $(x, y) \in R_{\mathbf{C}}$, then there exists $C \in \mathbf{C}$ such that $x, y \in C$. Hence $y \in C \subseteq \bigcup \{K \in \mathbf{C} | x \in K\} = I_{\mathbf{C}}(x)$.

(\impliedby): If $y \in I_{\mathbf{C}}(x) = \bigcup \{K \in \mathbf{C} | x \in K\}$, then there exists $C \in \mathbf{C}$ such that $x \in C$ and $y \in C$, which implies $(x, y) \in R_{\mathbf{C}}$.

As is well known, equivalence relations and partitions are determined by each other. Therefore, a question raises: what is the relationship between the type-1 characteristic matrix of a partition and the relational matrix of its corresponding equivalence relation?

Corollary 4. *Let R be an equivalence relation on U and M_R the relational matrix of R . Then $\Gamma(U/R) = M_R$.*

PROOF. We only need to prove $R_{U/R} = R$. In fact, it is straightforward since $I_{U/R}(x) = \bigcup \{K \in U/R | x \in K\} = [x]_R = \{y \in U | (x, y) \in R\}$.

The above corollary shows that the type-1 characteristic matrix of a partition coincides with the relational matrix of its corresponding equivalence relation. The following corollary considers another question: which covering blocks removed have no effect on type-1 characteristic matrix.

Corollary 5. *Let \mathbf{C} be a covering of U and $K \in \mathbf{C}$. If there exists $K' \in \mathbf{C} - \{K\}$ such that $K \subseteq K'$, then $\Gamma(\mathbf{C}) = \Gamma(\mathbf{C} - \{K\})$.*

3.3. Type-2 characteristic matrix of covering

The matrix representing a covering is a framework to study covering-based rough sets through matrices and it inherits essential information of the covering. This section constructs a new operation between boolean matrices to study covering-based rough sets in this framework.

Definition 7. *Let $A = (a_{ij})_{n \times m}$ and $B = (b_{ij})_{m \times p}$ be two boolean matrices. We define $C = A \odot B$ as follows: $C = (c_{ij})_{n \times p}$,*

$$c_{ij} = \bigwedge_{k=1}^m (b_{kj} - a_{ik} + 1).$$

It is worth noting that A and B are boolean matrices, however $A \odot B$ may not be a boolean matrix. The following counterexample indicates this argument.

Example 3. *Suppose $A = [0 \ \cdots \ 0]$ and $B = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, then $A \odot B = 2$.*

It is interesting that the new operation of a matrix representing a covering and its transpose is a boolean matrix.

Proposition 6. *Let \mathbf{C} be a covering of U and $M_{\mathbf{C}}$ a matrix representing \mathbf{C} . $M_{\mathbf{C}} \odot M_{\mathbf{C}}^T$ is a boolean matrix.*

PROOF. It is straightforward that $c_{ij} \in \{0, 1, 2\}$. We only need to prove $c_{ij} \neq 2$ for all $i, j \in \{1, \dots, n\}$.

$$M_{\mathbf{C}} \odot M_{\mathbf{C}}^T = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \odot (a_1^T, \dots, a_n^T) = \begin{pmatrix} a_1 \odot a_1^T & \cdots & a_1 \odot a_n^T \\ \vdots & & \vdots \\ a_n \odot a_1^T & \cdots & a_n \odot a_n^T \end{pmatrix} = (t_{ij})_{n \times n}.$$

$t_{ij} = a_i \odot a_j^T = \bigwedge_{k=1}^m (a_{kj} - a_{ik} + 1) = \bigwedge_{k=1}^m (a_{jk} - a_{ik} + 1)$. If $t_{ij} = 2$, then $\bigwedge_{k=1}^m (a_{jk} - a_{ik} + 1) = 2$, which implies $a_{jk} = 1$ and $a_{ik} = 0$ for all $k \in \{1, \dots, m\}$. In other words, $x_i \notin C_k$ for all $k \in \{1, \dots, m\}$, which is contradictory that \mathbf{C} is a covering of U . Therefore, $t_{ij} \in \{0, 1\}$, i.e., $M_{\mathbf{C}} \odot M_{\mathbf{C}}^T$ is a boolean matrix.

The following proposition points out that the new operation of any matrix representing a covering and its transpose is the same once a sort of elements of the universe is given.

Proposition 7. *Let \mathbf{C} be a covering of U and M_1, M_2 matrices representing \mathbf{C} . Then $M_1 \odot M_1^T = M_2 \odot M_2^T$.*

Definition 8. (Type-2 characteristic matrix of covering) Let \mathbf{C} be a covering of U . Then $M_{\mathbf{C}} \odot M_{\mathbf{C}}^T$ is called type-2 characteristic matrix of \mathbf{C} , denoted as $\Pi(\mathbf{C})$.

The following definition introduces an approach to generating a relation from a covering. This relation is closely connected with neighborhood-based rough sets.

Definition 9. (Relation induced by a covering [32]) Let \mathbf{C} be a covering of U . One can define the relation $R(\mathbf{C})$ on U as follows: for all $x, y \in U$,

$$(x, y) \in R(\mathbf{C}) \iff y \in N_{\mathbf{C}}(x).$$

The type-2 characteristic matrix of a covering is the relational matrix of the relation induced by neighborhoods.

Theorem 2. Let \mathbf{C} be a covering of U . Then $\Pi(\mathbf{C})$ is the relational matrix of $R(\mathbf{C})$.

PROOF. Denote $M_{\mathbf{C}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\Pi(\mathbf{C}) = M_{\mathbf{C}} \odot M_{\mathbf{C}}^T = (t_{ij})_{n \times n}$. If $t_{ij} = 1$,

then $\bigwedge_{k=1}^m (a_{jk} - a_{ik} + 1) = 1$, which implies if $a_{ik} = 1$, then $a_{jk} = 1$. In other words, if $x_i \in C_k$, then $x_j \in C_k$. Hence $x_j \in \bigcap \{K \in \mathbf{C} | x_i \in K\} = N_{\mathbf{C}}(x_i)$, i.e., $(x_i, x_j) \in R(\mathbf{C})$. If $t_{ij} = 0$, then $\bigwedge_{k=1}^m (a_{jk} - a_{ik} + 1) = 0$, which implies that there exists $k_0 \in \{1, \dots, m\}$ such that $a_{jk_0} = 0$ and $a_{ik_0} = 1$. In other words, $x_i \in C_{k_0}$ and $x_j \notin C_{k_0}$. Thus $x_j \notin C_{k_0} \supseteq \bigcap \{K \in \mathbf{C} | x_i \in K\} = N_{\mathbf{C}}(x_i)$, which implies $x_j \notin N_{\mathbf{C}}(x_i)$, i.e., $(x_i, x_j) \notin R(\mathbf{C})$. This completes the proof.

The following proposition considers a question: which covering blocks removed have no effect on type-2 characteristic matrix.

Proposition 8. Let \mathbf{C} be a covering of U and $K \in \mathbf{C}$. If K is reducible, then $\Pi(\mathbf{C}) = \Pi(\mathbf{C} - \{K\})$.

PROOF. It is straightforward since $N_{\mathbf{C}}(x) = N_{\mathbf{C}-\{K\}}(x)$ for all $x \in U$ if K is reducible in \mathbf{C} .

The above proposition presents that reducible elements of a covering removed have no effect type-2 characteristic matrix. It is natural that a covering and its reduct have the same type-2 characteristic matrix.

Proposition 9. Let \mathbf{C} be a covering of U . Then $\Pi(\mathbf{C}) = \Pi(\text{Reduct}(\mathbf{C}))$.

The neighborhood-based upper approximation can be represented by matrix representation. χ_Y is used to denote the characteristic function of Y in U ; in other words, for all $y \in U$, $\chi_Y(y) = 1$ if and only if $y \in Y$.

Theorem 3. Let \mathbf{C} be a covering of U . Then for all $X \subseteq U$,

$$\chi_{IH(X)} = \Pi(\mathbf{C}) \cdot \chi_X.$$

PROOF. Denote $M_{\mathbf{C}} \odot M_{\mathbf{C}}^T = (t_{ij})_{n \times n}$. If $X = \emptyset$, then $\chi_{IH(X)} = M_{\mathbf{C}} \odot M_{\mathbf{C}}^T \cdot [0, \dots, 0]^T = [0, \dots, 0]^T$, which implies $IH(X) = \emptyset$.

$$\begin{aligned}
& x_i \in IH(X) \\
& \Leftrightarrow \chi_{IH(X)}(x_i) = 1 \\
& \Leftrightarrow \bigvee_{k=1}^m (t_{ik} \wedge \chi_X(x_k)) = 1 \\
& \Leftrightarrow \exists k_0 \in \{1, \dots, m\} \text{ such that } t_{ik_0} = \chi_X(x_{k_0}) = 1 \\
& \Leftrightarrow x_{k_0} \in N_{\mathbf{C}}(x_i), x_{k_0} \in X \\
& \Leftrightarrow N_{\mathbf{C}}(x_i) \cap X \neq \emptyset.
\end{aligned}$$

Example 4. Let $U = \{a, b, c, d, e, f\}$ and $\mathbf{C} = \{K_1, K_2, K_3, K_4\}$ where $K_1 = \{a, b\}$, $K_2 = \{a, c, d\}$, $K_3 = \{a, b, c, d\}$ and $K_4 = \{d, e, f\}$. Then

$$\Pi(\mathbf{C}) = M_{\mathbf{C}} \odot M_{\mathbf{C}}^T = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

X	χ_X	$\Pi(\mathbf{C}) \cdot \chi_X$	$IH(X)$
$\{a\}$	$[1\ 0\ 0\ 0\ 0\ 0]^T$	$[1\ 1\ 1\ 0\ 0\ 0]^T$	$\{a, b, c\}$
$\{b, c\}$	$[0\ 1\ 1\ 0\ 0\ 0]^T$	$[0\ 1\ 1\ 0\ 0\ 0]^T$	$\{b, c\}$
$\{a, d, e\}$	$[1\ 0\ 0\ 1\ 1\ 0]^T$	$[1\ 1\ 1\ 1\ 1\ 1]^T$	$\{a, b, c, d, e, f\}$
$\{b, d, e, f\}$	$[0\ 1\ 0\ 1\ 1\ 1]^T$	$[0\ 1\ 1\ 1\ 1\ 1]^T$	$\{b, c, d, e, f\}$

Similarly, the neighborhood-based lower approximation operator is also represented by type-2 characteristic matrix.

Theorem 4. Let \mathbf{C} be a covering of U . Then for all $X \subseteq U$ and $X \neq \emptyset$,

$$\chi_{IL(X)} = \Pi(\mathbf{C}) \odot \chi_X.$$

PROOF. Denote $M_{\mathbf{C}} \odot M_{\mathbf{C}}^T = (t_{ij})_{n \times n}$.

$$\begin{aligned}
& x_i \in IL(X) \Leftrightarrow \chi_{IL(X)}(x_i) = 1 \\
& \Leftrightarrow \bigvee_{k=1}^m (\chi_X(x_k) - t_{ik} + 1) = 1 \\
& \Leftrightarrow \text{if } t_{ik} = 1, \text{ then } \chi_X(x_k) = 1 \\
& \Leftrightarrow \text{if } x_k \in N_{\mathbf{C}}(x_i), \text{ then } x_k \in X \\
& \Leftrightarrow N_{\mathbf{C}}(x_i) \subseteq X.
\end{aligned}$$

Example 5. As shown in Example 4, the following table is obtained:

X	χ_X	$\Pi(\mathbf{C}) \odot \chi_X$	$IL(X)$
$\{a\}$	$[1\ 0\ 0\ 0\ 0\ 0]^T$	$[1\ 0\ 0\ 0\ 0\ 0]^T$	$\{a\}$
$\{b, c\}$	$[0\ 1\ 1\ 0\ 0\ 0]^T$	$[0\ 0\ 0\ 0\ 0\ 0]^T$	\emptyset
$\{a, d, e\}$	$[1\ 0\ 0\ 1\ 1\ 0]^T$	$[1\ 0\ 0\ 1\ 0\ 0]^T$	$\{a, d\}$
$\{b, d, e, f\}$	$[0\ 1\ 0\ 1\ 1\ 1]^T$	$[0\ 0\ 0\ 1\ 1\ 1]^T$	$\{d, e, f\}$
$\{a, b, c, d, e\}$	$[1\ 1\ 1\ 1\ 1\ 0]^T$	$[1\ 1\ 1\ 1\ 0\ 0]^T$	$\{a, b, c, d\}$

Using the new operation, type-1 characteristic matrix is examined. Then the covering-based upper approximation operator is described.

Theorem 5. Let \mathbf{C} be a covering of U . Then for all $X \subseteq U$,

$$\chi_{SH(X)} = \Gamma(\mathbf{C}) \cdot \chi_X.$$

PROOF. We only need to prove $\{x \in U | I(x) \cap X \neq \emptyset\} = SH(X)$ for all $X \subseteq U$, where $I(x) = \bigcup \{K \in \mathbf{C} | x \in K\}$. For all $x \in SH(X)$, then there exists $K \in \mathbf{C}$ such that $K \cap X \neq \emptyset$. Then $K \cap X \subseteq I(x) \cap X \neq \emptyset$, which implies $x \in \{x \in U | I(x) \cap X \neq \emptyset\}$. Conversely, for all $x \notin \{x \in U | I(x) \cap X \neq \emptyset\}$, i.e., $I(x) \cap X = \emptyset$ which implies $I(x) \subseteq X^c$. Since $K \cap X \subseteq I(x) \cap X$ for all $x \in K$, then $K \cap X = \emptyset$. Hence $x \notin SH(X)$. This completes the proof.

Example 6. As shown in Example 4, the following table is presented:

X	χ_X	$\Gamma(\mathbf{C}) \cdot \chi_X$	$SH(X)$
$\{a\}$	$[1\ 0\ 0\ 0\ 0\ 0]^T$	$[1\ 1\ 1\ 1\ 0\ 0]^T$	$\{a, b, c, d\}$
$\{b, c\}$	$[0\ 1\ 1\ 0\ 0\ 0]^T$	$[1\ 1\ 1\ 1\ 0\ 0]^T$	$\{a, b, c, d\}$
$\{a, d, e\}$	$[1\ 0\ 0\ 1\ 1\ 0]^T$	$[1\ 1\ 1\ 1\ 1\ 1]^T$	$\{a, b, c, d, e, f\}$
$\{a, b, c, d\}$	$[1\ 1\ 1\ 1\ 0\ 0]^T$	$[1\ 1\ 1\ 1\ 1\ 1]^T$	$\{a, b, c, d, e, f\}$
$\{a, b, d, e, f\}$	$[1\ 1\ 0\ 1\ 1\ 1]^T$	$[1\ 1\ 1\ 1\ 1\ 1]^T$	$\{a, b, c, d, e, f\}$

Similarly, the covering-based lower approximation operator is represented using type-1 characteristic matrix and the new operation.

Theorem 6. Let \mathbf{C} be a covering of U . Then for all $X \subseteq U$,

$$\chi_{SL(X)} = \Gamma(\mathbf{C}) \odot \chi_X.$$

PROOF. We only need to prove $\{x \in U | I(x) \subseteq X\} = SL(X)$ for all $X \subseteq U$. In fact, it is straightforward.

Example 7. As shown in Example 4, the following table is exhibited:

X	χ_X	$\Gamma(\mathbf{C}) \odot \chi_X$	$SL(X)$
$\{a\}$	$[1\ 0\ 0\ 0\ 0\ 0]^T$	$[0\ 0\ 0\ 0\ 0\ 0]^T$	\emptyset
$\{a, b\}$	$[1\ 1\ 0\ 0\ 0\ 0]^T$	$[0\ 0\ 0\ 0\ 0\ 0]^T$	\emptyset
$\{d, e, f\}$	$[0\ 0\ 0\ 1\ 1\ 1]^T$	$[0\ 0\ 0\ 0\ 1\ 1]^T$	$\{e, f\}$
$\{a, b, c, d\}$	$[1\ 1\ 1\ 1\ 0\ 0]^T$	$[1\ 1\ 1\ 0\ 0\ 0]^T$	$\{a, b, c\}$
$\{a, b, d, e, f\}$	$[1\ 1\ 0\ 1\ 1\ 1]^T$	$[0\ 0\ 0\ 0\ 1\ 1]^T$	$\{e, f\}$

Theorem 7. Let \mathbf{C} be a covering of U . Then for all $X \subseteq U$,

$$\begin{aligned}\chi_{XH(X)} &= \Pi(\mathbf{C})^T \cdot \Pi(\mathbf{C}) \cdot \chi_X, \\ \chi_{XL(X)} &= \Pi(\mathbf{C})^T \cdot \Pi(\mathbf{C}) \odot \chi_X.\end{aligned}$$

PROOF. For a covering \mathbf{C} of U , we construct a special covering $Cov(\mathbf{C}) = \{N(x) | x \in U\}$ induced \mathbf{C} . Then $XH_{\mathbf{C}}(X) = SH_{Cov(\mathbf{C})}(X)$ and $XL_{\mathbf{C}}(X) = SL_{Cov(\mathbf{C})}(X)$ for all $X \subseteq U$. Since $\Pi(\mathbf{C})^T$ is a matrix representing $Cov(\mathbf{C})$, according to Theorems 5 and 6, $\chi_{XH(X)} = \Pi(\mathbf{C})^T \cdot \Pi(\mathbf{C}) \cdot \chi_X$ and $\chi_{XL(X)} = \Pi(\mathbf{C})^T \cdot \Pi(\mathbf{C}) \odot \chi_X$.

Example 8. As shown in Example 4, the following two tables are revealed:

Y	χ_Y	$\Pi(\mathbf{C})^T \cdot \Pi(\mathbf{C}) \cdot \chi_Y$	$XH(Y)$
$\{a\}$	$[1\ 0\ 0\ 0\ 0\ 0]^T$	$[1\ 1\ 1\ 1\ 0\ 0]^T$	$\{a, b, c, d\}$
$\{a, b\}$	$[1\ 1\ 0\ 0\ 0\ 0]^T$	$[1\ 1\ 1\ 1\ 0\ 0]^T$	$\{a, b, c, d\}$
$\{a, b, c\}$	$[1\ 1\ 1\ 0\ 0\ 0]^T$	$[1\ 1\ 1\ 1\ 0\ 0]^T$	$\{a, b, c, d\}$
$\{d, e, f\}$	$[0\ 0\ 0\ 1\ 1\ 1]^T$	$[1\ 0\ 1\ 1\ 1\ 1]^T$	$\{a, c, d, e, f\}$
$\{a, d, e, f\}$	$[1\ 0\ 0\ 1\ 1\ 1]^T$	$[1\ 1\ 1\ 1\ 1\ 1]^T$	$\{a, b, c, d, e, f\}$

Y	χ_Y	$\Pi(\mathbf{C})^T \cdot \Pi(\mathbf{C}) \odot \chi_Y$	$XL(Y)$
$\{a\}$	$[1\ 0\ 0\ 0\ 0\ 0]^T$	$[0\ 0\ 0\ 0\ 0\ 0]^T$	\emptyset
$\{a, b\}$	$[1\ 1\ 0\ 0\ 0\ 0]^T$	$[0\ 1\ 0\ 0\ 0\ 0]^T$	$\{b\}$
$\{a, b, c\}$	$[1\ 1\ 1\ 0\ 0\ 0]^T$	$[0\ 1\ 0\ 0\ 0\ 0]^T$	$\{b\}$
$\{a, b, c, d\}$	$[1\ 1\ 1\ 1\ 0\ 0]^T$	$[1\ 1\ 1\ 0\ 0\ 0]^T$	$\{a, b, c\}$
$\{a, b, d, e, f\}$	$[1\ 1\ 0\ 1\ 1\ 1]^T$	$[0\ 1\ 0\ 0\ 1\ 1]^T$	$\{b, e, f\}$
$\{a, b, c, d, e, f\}$	$[1\ 1\ 1\ 1\ 1\ 1]^T$	$[1\ 1\ 1\ 1\ 1\ 1]^T$	$\{a, b, c, d, e, f\}$

4. Boolean matrix decomposition using covering-based rough sets

In this section, we present a sufficient and necessary condition for a boolean matrix to decompose into the boolean product of another boolean matrix and its transpose, i.e., $B = A \cdot A^T$, where $B \in \{0, 1\}^{n \times n}$ and $A \in \{0, 1\}^{n \times m}$. Here $\{0, 1\}^{n \times m}$ denotes the family of all boolean matrices $C = (C_{ij})_{n \times m}$.

Proposition 10. *Let \mathbf{C} be a covering of U and $M_{\mathbf{C}}$ a matrix representing \mathbf{C} . Then $M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T$ is symmetric and $(M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T)_{ii} = 1$.*

For all $A, B, C \in \{0, 1\}^{n \times m}$, if $A_{ij} = B_{ij} \vee C_{ij}$ for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, then we denote $A = B \vee C$ and we say that A is the union of B and C . If $A_{ij} \leq B_{ij}$ for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, then we denote $A \leq B$. Obviously, $A = B$ if and only if $A \leq B$ and $B \leq A$.

In the following proposition, we break the characteristic matrix of a covering into the union of some “small” characteristic matrices.

Proposition 11. *Let \mathbf{C} be a covering of $U = \{x_1, \dots, x_n\}$. Then*

$$M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T = \vee_{K \in \mathbf{C}} (M_{\{K\}} \cdot M_{\{K\}}^T).$$

PROOF. It is straightforward that $\vee_{K \in \mathbf{C}} (M_{\{K\}} \cdot M_{\{K\}}^T) \leq M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T$. If $(M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T)_{ij} = 1$, then according to Proposition 5, $x_i \in I_{\mathbf{C}}(x_j)$. Hence there exists $K \in \mathbf{C}$ such that $x_i, x_j \in K$, which implies $(M_{\{K\}} \cdot M_{\{K\}}^T)_{ij} = 1$. This proves that $M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T \leq \vee_{K \in \mathbf{C}} (M_{\{K\}} \cdot M_{\{K\}}^T)$. To sum up, we prove $M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T = \vee_{K \in \mathbf{C}} (M_{\{K\}} \cdot M_{\{K\}}^T)$.

In fact, the characteristic matrix of a covering can be represented by the characteristic functions of covering blocks.

Corollary 6. *Let \mathbf{C} be a covering of U . Then*

$$M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T = \bigvee_{K \in \mathbf{C}} (\chi_K \cdot \chi_K^T).$$

Inspired by the “small” characteristic matrix of a covering block, we define the sub-formula, which serves as a foundation for designing an algorithm for an optimal boolean matrix decomposition.

Definition 10. Let $U = \{x_1, \dots, x_n\}$ be a universe and $A \in \{0, 1\}^{n \times n}$. A is called a sub-formula on U if there exists $X \subseteq U$ such that $A = M_{\{X\}} \cdot M_{\{X\}}^T$.

Example 9. Let $A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Then A_1 and A_2 are sub-formulas on $U = \{x_1, x_2, x_3\}$ since there exist $X_1 = \{x_1, x_3\}$ and $X_2 = \{x_1, x_2, x_3\}$ such that $A_1 = M_{\{X_1\}} \cdot M_{\{X_1\}}^T$ and $A_2 = M_{\{X_2\}} \cdot M_{\{X_2\}}^T$.

In the following theorem, we present a sufficient and necessary condition for a square boolean matrix to decompose into the boolean product of another boolean matrix and its transpose.

Theorem 8. Let $B \in \{0, 1\}^{n \times n}$. Then there exists $A \in \{0, 1\}^{n \times m}$ such that $B = A \cdot A^T$ iff $(B = B^T) \wedge (\forall i, j \in \{1, \dots, n\}, B_{ij} = 1 \Rightarrow B_{ii} = 1)$.

PROOF. (\Rightarrow): On one hand, $B^T = (A \cdot A^T)^T = (A^T)^T \cdot A^T = A \cdot A^T = B$. On the

other hand, we suppose $A = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix}$, where a_i is a m -dimensional column vector.

$\forall i, j \in \{1, \dots, n\}$, if $B_{ij} = 1$, then $a_i^T \cdot a_j = 1$, i.e., there exists $h \in \{1, \dots, m\}$ such that $a_{ih} = a_{jh} = 1$. Hence $B_{ii} = a_i^T \cdot a_i = 1$ and $B_{jj} = a_j^T \cdot a_j = 1$.

(\Leftarrow): Suppose that A_1, \dots, A_m is all the sub-formulas that satisfy $A_i \leq B$, i.e., $A_i \vee B = B$, $i \in \{1, \dots, m\}$. If $(B = B^T) \wedge (\forall i, j \in \{1, \dots, n\}, B_{ij} = 1 \Rightarrow B_{ii} = 1)$, then $B = \bigvee_i A_i$. Since A_i is a sub-formula on $U = \{x_1, \dots, x_n\}$, there exists a unique K_i such that $A_i = M_{\{K_i\}} \cdot M_{\{K_i\}}^T$. Hence $B = \bigvee_i A_i = \bigvee_i (M_{\{K_i\}} \cdot M_{\{K_i\}}^T)$. According to Proposition 11, $B = M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T$ where $\mathbf{C} = \{K_1, \dots, K_m\}$, which implies $B = A \cdot A^T$ where $A = M_{\mathbf{C}}$.

In applications, $\min\{k | B = A \cdot A^T, A \in \{0, 1\}^{n \times k}\}$ has some special meaning. For instance, in role mining, it represents the minimal number of roles [20]. If $B = A \cdot A^T$ where A has a minimal column, then it is an optimal decomposition. The following reveals the special meaning of an optimal decomposition in covering-based rough sets. For this reason, the reducible element is introduced.

Definition 11. (Reducible element [28, 33]) Let \mathbf{C} be a covering of U and $K \in \mathbf{C}$. If K can be expressed as a union of some elements in $\mathbf{C} - \{K\}$, then K is called a union-reducible element of \mathbf{C} ; otherwise, it is called a union-irreducible element. Similarly, if K can be expressed as an intersection of some elements in $\mathbf{C} - \{K\}$, then K is called an intersection-reducible element of \mathbf{C} ; otherwise, it is called an intersection-irreducible element.

Inspired by the reducible element, we define the notion of general intersection-reducible element, which is similar to the relative covering reduct proposed in literature [6].

Definition 12. Let \mathbf{C} be a covering of U and $K \in \mathbf{C}$. If there exists $K' \in \mathbf{C}$ such that $K \subseteq K'$, then K is called a general intersection-reducible element of \mathbf{C} ; otherwise, it is called general intersection-irreducible.

Note that the above definition can be extended to any family of subsets of a set. Hence in the rest of this paper, this notion may be used to deal with a family of subsets of a set without additional definition.

In fact, the family of all general intersection-reducible elements of \mathbf{C} is unique, and we call it the general intersection-reduct of \mathbf{C} , and denote it as $GIR(\mathbf{C})$. The following theorem explores the relationship between an optimal boolean matrix decomposition and the general intersection-reduct.

Theorem 9. Let $B \in \{0, 1\}^{n \times n}$ where $B = B^T$ and $\forall i, j \in \{1, \dots, n\}, B_{ij} = 1 \Rightarrow B_{ji} = 1$. Then $B = M_{GIR(\mathbf{C}_B)} \cdot M_{GIR(\mathbf{C}_B)}^T$ is an optimal decomposition of B , where $\mathbf{C}_B = \{K | \exists A \text{ s.t. } (A \leq B) \wedge (M_{\{K\}} \cdot M_{\{K\}}^T = A)\} - \{\emptyset\}$.

Theorem 9 shows that finding an optimal boolean matrix decomposition is equivalently to find the general intersection-reduct of a covering. An example is provided to illustrate this interesting transformation.

Example 10. Let $B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then the maximal sub-formulas containing in B are $A_1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. We suppose $U = \{x_1, \dots, x_5\}$ and $K_1 = \{x_1, x_2, x_4\}$ and $K_2 = \{x_3, x_4\}$, then $A_1 = M_{\{K_1\}} \cdot M_{\{K_1\}}^T$ and $A_2 = M_{\{K_2\}} \cdot M_{\{K_2\}}^T$. Hence $B = A_1 \vee A_2 = (M_{\{K_1\}} \cdot M_{\{K_1\}}^T) \vee (M_{\{K_2\}} \cdot M_{\{K_2\}}^T) = M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T$ where $\mathbf{C} = \{K_1, K_2\}$.

The following algorithm shows how to obtain an optimal boolean matrix decomposition using covering-based rough sets.

Algorithm 1 An algorithm for optimal boolean matrix decomposition

Input: $B \in \{0, 1\}^{n \times n}$

Output: $A \in \{0, 1\}^{n \times m}$ with the minimal column

- 1: Denote $U = \{x_1, \dots, x_n\}$;
 - 2: Compute all maximal sub-formulas containing in B , and denote as A_1, \dots, A_m ;
 - 3: **if** $B = \bigvee_{i=1}^m A_i$ **then**
 - 4: Compute K_i such that $A_i = M_{\{K_i\}} \cdot M_{\{K_i\}}^T$ for $i \in \{1, \dots, m\}$;
 - 5: Return $A = [\chi_{K_1}, \dots, \chi_{K_m}]$;
 - 6: // $A \cdot A^T$ is an optimal boolean matrix decomposition of B ;
 - 7: **else**
 - 8: Return \emptyset ; // there does not exist $A \in \{0, 1\}^{n \times m}$ such that $B = A \cdot A^T$;
 - 9: **end if**
-

5. Axiomatization of covering approximation operators using boolean matrices

Axiomatization of covering-based rough sets has attracted much research interest [14, 30, 35]. However, those works are mainly conducted from the viewpoints of set theory and operator theory. As a well-known quantitative tool, boolean matrices are used to axiomatize the upper approximation operators of covering-based rough sets in this paper.

Let $U = \{x_1, \dots, x_n\}$ and an operator $f : 2^U \rightarrow 2^U$. We denote $A_f = [\chi_{f(e_1)}, \dots, \chi_{f(e_n)}]^T$ where $e_i = \underbrace{[\dots, 0, 1, 0, \dots]}_{i\text{-th}}$.

Theorem 10. *Let $H : 2^U \rightarrow 2^U$ be an operator. Then there exists a covering \mathbf{C} such that $H = SH_{\mathbf{C}}$ iff $A_H^T = A_H$ and $(A_H)_{ii} = 1$ for all $i \in \{1, \dots, n\}$.*

PROOF. (\implies): Since there exists a covering \mathbf{C} such that $H = SH_{\mathbf{C}}$, $A_H = \Gamma(\mathbf{C}) = M_{\mathbf{C}} \cdot M_{\mathbf{C}}^T$. According to Proposition 4, $(A_H)_{ii} = 1$ for all $i \in \{1, \dots, n\}$. Additionally, $A_H^T = A_H$ is straightforward.

(\impliedby): Since $A_H^T = A_H$ and $(A_H)_{ii} = 1$ for all $i \in \{1, \dots, n\}$, according to Theorem 8, there exists $B \in \{0, 1\}^{n \times m}$ such that $A_H = B \cdot B^T$. Specifically, we suppose B is the minimal decomposition of A_H and $B = [B_1, \dots, B_m]$, then we construct a covering $\mathbf{C} = K_1, \dots, K_m$ satisfying $B_i = \chi_{K_i}$ for all $i \in \{1, \dots, m\}$. Hence it is straightforward that $H = SH_{\mathbf{C}}$.

Theorem 10 shows a sufficient and necessary condition for an operator to be the second upper approximation operator with respect to a covering using boolean matrices. The following corollary reveals the close connection between covering-based rough sets and generalized rough sets based on relations.

Corollary 7. *Let $H : 2^U \rightarrow 2^U$ be an operator. Then there exists a covering \mathbf{C} such that $H = SH_{\mathbf{C}}$ iff A_H is the relational matrix of a reflexive and symmetric relation.*

Corollary 7 presents that the second upper approximation operator with respect to a covering is determined by a reflexive and symmetric relation. The following theorem explores the relationship between the fifth upper approximation operator of covering-based rough sets and generalized rough sets based on relations.

Theorem 11. *Let $H : 2^U \rightarrow 2^U$ be an operator. Then there exists a covering \mathbf{C} such that $H = IH_{\mathbf{C}}$ iff $(A_H)^2 = A_H$ and $(A_H)_{ii} = 1$ for all $i \in \{1, \dots, n\}$.*

Theorem 11 exhibits a sufficient and necessary condition for an operator to be the fifth upper approximation one with respect to a covering using boolean matrices.

Corollary 8. *Let $H : 2^U \rightarrow 2^U$ be an operator. Then there exists a covering \mathbf{C} such that $H = IH_{\mathbf{C}}$ iff A_H is the relational matrix of a reflexive and transitive relation.*

Corollary 8 points out that the fifth upper approximation operator with respect to a covering is determined by a reflexive and transitive relation.

Theorem 12. *Let $H : 2^U \rightarrow 2^U$ be an operator. Then there exists a covering \mathbf{C} such that $H = XH_{\mathbf{C}}$ iff there exists $B \in \{0, 1\}^{n \times n}$ such that $A_H = B \cdot B^T$ where $B^2 = B$ and $(B)_{ii} = 1$ for all $i \in \{1, \dots, n\}$.*

PROOF. According to Theorem 7, it is straightforward.

Theorem 12 presents an axiom of the sixth upper approximation operator with respect to a covering from the viewpoint of boolean matrices.

Corollary 9. *Let $H : 2^U \rightarrow 2^U$ be an operator. Then there exists a covering \mathbf{C} such that $H = XH_{\mathbf{C}}$ iff there exists a reflexive and transitive relation (B its relational matrix) such that $A_H = B \cdot B^T$.*

Corollary 9 indicates that the sixth upper approximation operator with respect to a covering is corresponded with a reflexive and transitive relation.

6. Conclusions

In this paper, we define two types of characteristic matrices of a covering, and then use them to represent three types of widely used covering approximation operators. Through matrix representations of covering approximation operators, we provide a sufficient and necessary condition a boolean matrix to decompose into the boolean product of another boolean matrix and its transpose. And then we design an algorithm for this boolean matrix decomposition. Finally, we axiomatize these three types of covering approximation operators using boolean matrices. This work may open up a new view to study covering-based rough sets.

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